1. COMPLEX NETWORKS

A graph is a set of points or nodes joined by lines called edges. For example, a network of computers is a graph with the computers being the nodes and the connecting wires being the edges. The Internet forms a large graph, as do the neurons in the brain. These graphs are large and complicated with dynamic components. They are called complex networks and the study of complex networks is called network science. In the RAMMP summer REU project students will first learn some of the theoretical, algorithmic, and statistical methods required to analyze complex networks from the physical sciences (climate, molecular structure), the biological sciences (connectome, epidemics), the social sciences and humanities (social networks), etc. Students will then undertake an undergraduate research project that consist of two parts: first students will analyze several known large networks using techniques they learned to see if there is any new insight to present about known networks; then they will model an original real-world problem using networks and will create at least one new technique for analyzing their large network.

The connectome of a worm [Jab12].

2. THE BANACH-TARSKI PARADOX

A key step in the Banach-Tarski Paradox is to find two rotations R, S of three-dimensional space where if

\[ A_1A_2\cdots A_r = 1 \]

are elements from \( \{R, R^{-1}, S, S^{-1}\} \), then there must be some \( i \) with \( A_i \) and \( A_{i+1} \) are inverse of one another. In group theory language, this means there is a free group of rank two inside the group of rotations.

**Problem.** Learn how to prove the above statement, with the goal of trying to see if it is possible to add the restriction that all the the matrices R, S only have rational entries.
3. TILING BILLIARDS

Tiling billiards concerns the behavior of light rays in a tiling which refract whenever they move through two adjacent tiles as if the index of refraction is negative one. This means that the angle of refraction is the negative of the angle of incidence, where these angles are measured from the normal vector to the edge the trajectory is passing through; see the left figure below:

Several types of behavior are possible for light rays (also called trajectories) in tiling billiards. They can close up, returning to the starting point traveling the same direction. In this case the ray is called periodic. If the tiling has translational symmetries, a light ray can be preserved by a translation of the tiling. In this case the ray is called drift periodic.

Some types of tilings have been studied, including reflective tilings [DDRSL18], triangle tilings [BSDFI18] [HPR18], and the trihexagonal tiling [DH18].

Problem. Study tiling billiards in some new examples of tilings. For example, the semi-regular tilings (such as the one showed below left) and the Pythagorean tiling (below right) have not been studied. How common are periodic and drift-periodic trajectories in these tilings?

4. PROGRAMMING GROUPS INTO A COMPUTER: THE ABSTRACT COMMENSURATOR

At the interface of computer science and group theory lies the question of whether groups can be fully understood through a computer. Matrices are an important tool for these constructions, but sometimes there is no way to represent a group using matrices. In those situations, one has to turn to other methods, such as the abstract commensurator.

Problem. Show that there is a program fully encoding any finitely generated subgroup of the abstract commensurator of a free group. Can one extend this to the entire abstract commensurator of a free group?
Solving this problem gives algorithms for representing any group in terms of a graph, called the *Cayley graph* of the group.

5. **INFINITE RATIONAL IETS**

An *interval exchange map (IET)* starts with an interval $I \subset \mathbb{R}$ and cuts it into subintervals, then rearranges the subintervals by translation to obtain a bijection $I \to I$. An IET is called *infinite* if $I$ is cut into infinitely many subintervals.

In the article [HRR18], a fairly simple IET $T : [0, 1) \to [0, 1)$ is defined by cutting $[0, 1)$ at each point $\frac{1}{n}$ for $n \in \mathbb{N}$, and then reversing the order of the subintervals. This IET is depicted below:

![Diagram of an infinite IET](image)

Most points tend to be *periodic* under this map, meaning that the sequence of points $x, T(x), T \circ T(x), T \circ T \circ T(x), \ldots$ repeats. (Here *most* means the set of periodic points is a countable union of intervals whose total length equals one.) Nonetheless, it was shown that a small fractal set of points is not periodic.

This IET is special for a number of reasons. Mainly, all the intervals are translated by an rational amount (e.g., the red interval above is translated by $\frac{1}{2}$, the blue by $-\frac{1}{6}$, $\ldots$). We call such an IET *rational*.

Recently, student Anna Tao found another rational infinite IET which displays similar properties. Her IET cuts up the interval $[0, e)$ into subintervals of size $\frac{1}{n!}$ for $n \in \mathbb{N}$.

**Problem.** Find new examples of rational infinite IETs which exhibit the property that most points are periodic. Perhaps a sum formula for $\pi$ or $\sqrt{2}$ can be used to produce such an IET. Investigate other possible behaviors of these maps.

### References